Internal Report: Duality Controllability-Observability on Fork-Attribution Fluid Petri Nets

C. R. Vázquez [*]J.L García-Malacara, A. Ramírez-Treviño [†]

April 28, 2014

Abstract

The work is concerned with duality of controllability and observability of Timed continuous Petri nets ($TCPNs$). According to the duality definition presented in Hack (1972), Fork-Attribution (FA) $TCPN$ are closed under the duality transformation. Based on this property, in this work it is showed that the controllability of a $FA - TCPN$ can be explained in observability terms of its dual. In fact, the rank of the controllability matrix of a $FA - TCPN$ and the rank of the observability matrix of its dual are related. The analysis here presented can be used to translate previous results of observability analysis to controllability and vice versa.

1 Introduction

Nowadays the scientific community of Discrete Event Systems (DES) have extended the discrete Petri nets (PN) to the fluidized Petri nets (David and Alla (2010); Silva et al. (2011)) with the aim to model, analyse and control highly populated systems. This tool had been used to model and analyse biological systems, network traffic systems among many others (Júlvez and Boel (2010); Ross-León et. al. (2010)). A lot of attention has been paid to properties such as controllability (Mahulea et al. (2008); Vázquez and Ramírez (2012); Vázquez et. al. (2013)) and observability (Aguayo et al. (2011); Silva et al. (2011)), since they allow to implement state feedback controllers (Lefebvre (1999); Kara et al. (2009); Mahulea et al. (2008,b)).

In the works reported in the literature, the controllability has been addressed analysing the solution of the state equation. In this case, the system is controllable if the span of the generated space from the equilibria points includes the set of all equilibria points. The observability has been addressed from another perspective, where the net structure is used to determine when the observability space is complete. Now, this work explores the existence of a common framework to analyse both properties: observability and controllability. It is performed by building a bridge between the controllability and observability based on the concept of duality.

[*]CUValles, Universidad de Guadalajara Ameca Jalisco, México

[†]Unidad Guadalajara, Cinvestav del IPN, Guadalajara Jalisco, México

In linear systems (LS), the concept of duality is broadly used (see for example Chen (1984)), and the observability and controllability can be seen as dual concepts. For instance, the observability property in a LS can be determined studying the controllability of the dual system and vice versa. These results are important because when a $TCPN$ does not have join transitions it can be modelled as a linear system.

In this work we adopt the definition of duality in discrete Petri nets presented in Hack (1972), where the dual net is obtained by changing each place of the net to a transition and each transition to a place, and the direction of arcs are reversed. A PN that is join free and its dual is also join free, belongs to the class of Fork-Attribution. Therefore this paper focus on $FA-TCPN$.

The contribution of the paper regards the relation between the controllability property of a net and the observability property of its dual. In particular, it is showed that the controllability matrix of a $FA-TCPN$ is related to the observability matrix of its dual. Using this relation, different controllability and observability approaches can be merged. For instance, it is shown that the loss of observability introduced by an attribution in a net is reflected in the loss of controllability in its dual, which is explained by a linear dependence on the flows of certain transitions introduced by a fork (the dual node of the attribution).

This work is organized as follows. Section 2 presents basic concepts related with $TCPN$ and duality. Next, section 3 provides an explanation for the loss of controllability due to forks in FA nets. Section 4 presents the duality between controllability and observability in FA nets. Section 5 exploits the duality result presented in section 4 to extend the controllability result in section 3 to observability in FA nets. Finally the conclusions and future work is presented.

2 Basic Concepts

We assume that the reader is familiar with Petri nets (see, for instance, Silva (1993), David and Alla (2010)). The set of the input (output) nodes of v is denoted as $^\bullet v$ ($v^\bullet$). In the sequel, given a vector $\mathbf{a}$, a_j denotes the j-th entry of $\mathbf{a}$. Similarly, $[\mathbf{A}]_{i,j}$ denotes the entry at the i-th row and j-th column of matrix $\mathbf{A}$.

Definition. A *continuous PN* system is a pair $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ where $\mathcal{N}$ is a P/T net and $\mathbf{m}_0 \in \mathbb{R}_{\geq 0}^{|P|}$ is the initial marking. That is, P is a finite set of places, T is a finite set of transitions with $P \cap T = \emptyset$, $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, *pre- and post- incidence matrices.* In continuous PNs, the transitions can fire at real rates, thus the marking $\mathbf{m} \in \mathbb{R}_{\geq 0}^{|P|}$ is not forced to be integer. Instead, a transition t_i is *enabled* at $\mathbf{m}$ iff for every $p_j \in {^\bullet t_i}$, $\mathbf{m}[p_j] > 0$; and its *enabling degree* is $enab(t_i, \mathbf{m}) = \min_{p_j \in {^\bullet t_i}} \{\mathbf{m}[p_j]/\mathbf{Pre}[p_j, t_i]\}$. The firing of t_i in a certain amount $\alpha \leq enab(t_i, \mathbf{m})$ leads to a new marking $\mathbf{m'} = \mathbf{m} + \alpha \cdot \mathbf{C}[P, t_i]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the *incidence (token-flow) matrix.*

As in discrete systems, a column vector $\mathbf{y}$ s.t. $\mathbf{y}^T \cdot \mathbf{C} = \mathbf{0}$ ($\mathbf{x}$ s.t. $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$) is called *P-flow* (*T-flow*). When they are nonnegative, they are called *P-* and *T-semiflows.* Here, it is assumed that $\mathbf{m}_0$ marks all P-semiflows (otherwise, all the transitions in the corresponding P-component will be non-live). Matrix $\mathbf{B}_y$ denotes a basis of $P-flows$. If there exists $\mathbf{y} > \mathbf{0}$ s.t. $\mathbf{y}^T \cdot \mathbf{C} = \mathbf{0}$, the net is said to be *conservative*, and if there exists $\mathbf{x} > \mathbf{0}$ s.t. $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ the net is said to be *consistent.*

Definition. A *timed continuous Petri net (TCPN)* is a time-driven continuous-state system described by the tuple $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$, where $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ is a *continuous* PN and the vector $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$ represents the transitions rates. Transitions fire according to certain speed, which generally is a function of the rates and the instantaneous marking. Under *infinite server semantics* the *flow* (the firing speed, denoted as $\mathbf{f}(\mathbf{m})$) through a transition t_i is defined as the product of the rate, λ_i, and $enab(t_i, \mathbf{m})$, the instantaneous enabling of the transition, i.e., $f_i(\mathbf{m}) = \lambda_i \cdot enab(t_i, \mathbf{m}) = \lambda_i \cdot \min_{p \in {}^\bullet t_i} \{\mathbf{m}[p]/\mathbf{Pre}[p, t_i]\}$.

The enabling degree vector $\mathbf{enab}(\mathbf{m}) \in \mathbb{R}^{|T|}_{\geq 0}$ is defined s.t. $enab_i(\mathbf{m}) = enab(t_i, \mathbf{m})$. For the enabling degree to be well defined, every transition must have at least one input place. This will be assumed in the sequel.

The "min" in the flow definition leads to the concept of *configurations*: a *configuration* is a set of pairs $\mathcal{C} = \{(t_1, p^1), (t_2, p^2), ..., (t_{|T|}, p^{|T|})\}$ where $\forall t_k \in T$, $p^k \in {}^\bullet t_k$ is a place that, for a marking $\mathbf{m} \in \mathbb{R}^{|P|}_{\geq 0}$, provides the minimum ratio $\mathbf{m}(p^k)/\mathbf{Pre}(p^k, t_j)$. In such case, it is said that p^k *constrains* t_k. An upper bound for the number of configurations is $\prod_{t \in T} |{}^\bullet t|$.

For each possible configuration $\mathcal{C}_i$, a configuration matrix of dimension $|T| \times |P|$ is defined, denoted as $\mathbf{\Pi}_i$, as:

$$
\forall j \in \{1, .., |T|\}, k \in \{1, .., |P|\}
$$
$$
[\mathbf{\Pi}_i]_{j,k} = \begin{cases} \frac{1}{[\mathbf{Pre}]_{k,j}} & \text{if } p_k \text{ is constraining } t_j \text{ in } \mathcal{C}_i \\ 0 & \text{otherwise} \end{cases}
$$

The set of all configuration matrices of the net system is denoted as $\{\mathbf{\Pi}\}$. Given a marking $\mathbf{m}$, the configuration operator $\mathbf{\Pi}(\mathbf{m})$ is defined as: $\mathbf{\Pi}(\mathbf{m}) = \mathbf{\Pi}_i$ where $\mathcal{C}_i$ is associated to $\mathbf{m}$, i.e., if $\mathbf{\Pi}_i \mathbf{m} = \mathbf{enab}(\mathbf{m})$. If $\mathbf{m}$ can be associated to more than one configuration, i.e., if $\exists \mathbf{\Pi}_i, \mathbf{\Pi}_j$ s.t. $\mathbf{\Pi}_i \mathbf{m} = \mathbf{\Pi}_j \mathbf{m} = \mathbf{enab}(\mathbf{m})$, and a particular configuration is not specified in the context, then any of them can be taken (for convention, let us take the one with the lowest index). Then, the flow through the transitions can be written as the vector $\mathbf{f}(\mathbf{m}) = \mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m})\mathbf{m}$, where $\mathbf{\Lambda}$ is a diagonal matrix whose elements are those of $\boldsymbol{\lambda}$.

The set of markings that agree with P-flows (formally stated, $\{\mathbf{m} \in \mathbb{R}^{|P|}_{>0} | \mathbf{B}_y^T \mathbf{m} = \mathbf{B}_y^T \mathbf{m}_0\}$) can be partitioned (except on the borders) into convex subsets of markings associated to different configurations:

Definition. For each configuration $\mathcal{C}_i$ (equivalently, for each value $\mathbf{\Pi}_i$ that the configuration matrix can take), a marking *region* is defined as the set $\Re_i = \{\mathbf{m} \in \mathbb{R}^{|P|}_{>0} | \mathbf{B}_y^T \mathbf{m} = \mathbf{B}_y^T \mathbf{m}_0, \, \mathbf{\Pi}_i \mathbf{m} \leq \mathbf{\Pi}_j \mathbf{m}, \forall \mathbf{\Pi}_j \in \{\mathbf{\Pi}\}\}$.

Definition. The control vector $\mathbf{u} \in \mathbb{R}^{|T|}$ is defined s.t. u_i represents the control action on t_i. The effective flow through a controlled transition is given by: $w_i(\mathbf{m}) = f_i - u_i$, where $0 \leq u_i \leq \lambda_i \cdot enab_i(\mathbf{m})$.
Transitions in which a control action can be applied are called *controllable*. The set of all controllable transitions is denoted by T_c, and the set of uncontrollable transitions is $T_{nc} = T \setminus T_c$. If $t_i \in T_{nc}$ then u_i must be null.

Therefore, the behavior of a TCPN forced system is described by the state equation:

$$
\begin{array}{c} \overset{\bullet}{\mathbf{m}} = \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m})\mathbf{m} - \mathbf{C}\mathbf{u} \\ \mathbf{0} \leq \mathbf{u} \leq \mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m})\mathbf{m} \end{array} \tag{1}
$$

A control action that fulfills the required constraints, i.e., $\forall t_i \in T_{nc}\ u_i = 0$ and $\mathbf{0} \leq \mathbf{u} \leq \mathbf{\Lambda\Pi(m)m}$, is called *suitably bounded* (s.b.). Note that if an input $\mathbf{u}$ is not s.b. at some marking $\mathbf{m}$ then it cannot be applied.

The controllability matrix of a $TCPN$ in a given configuration $\mathcal{C}_i$ is defined as

$$Cont = \left[(\mathbf{C}\,[T_c])\,, (\mathbf{C\Lambda\Pi}_i) \cdot (\mathbf{C}\,[T_c])\,, ..., (\mathbf{C\Lambda\Pi}_i)^{n-1} \cdot (\mathbf{C}\,[T_c])\right]$$

.

The output matrix of a system is $\mathbf{O}^T$, built with elementary row vectors. This matrix is a selector for the measured places of the net.

The observability matrix of a $TCPN$ in a given configuration $\mathcal{C}_i$ is built as

$$Obs = \begin{bmatrix} \mathbf{O}^T \\ \mathbf{O}^T \cdot \mathbf{C\Lambda\Pi}_i \\ \vdots \\ \mathbf{O}^T \cdot (\mathbf{C\Lambda\Pi}_i)^{n-1} \end{bmatrix}$$

.

A net is *strongly connected* iff for every pair of nodes x and y, there is a path leading from x to y. Two transitions t and t' are said to be in conflict if $^\bullet t \cap ^\bullet t' \neq \emptyset$. Two transitions t and t' are said to be in topological equal conflict relation if $\exists \gamma > 0$ s.t. $Pre[P,t] = \gamma Pre[P,t']$.

Subclasses of nets are defined according to their structure:
1) $\mathcal{N}$ is Choice-free (CF) iff $\forall p \in P : |p^\bullet| \leq 1$.
2) $\mathcal{N}$ is Join-free (JF) iff $\forall t \in T : |^\bullet t| \leq 1$.
3) $\mathcal{N}$ is Fork-Attribution (FA) iff it is CF and JF.

Now the definition of the reverse-dual Hack (1972) of a discrete Petri net is recalled, hereinafter called dual.

Definition. A dual of a primal PN $\langle\mathcal{N}\rangle$ is another PN $\langle\mathcal{N}^{\mathcal{T}}\rangle$ where the sets of places and transitions are interchanged and the direction of arcs are reversed.

Proposition 1. *Let N be a FA $TCPN$ where ending places are not allowed. N is named the primal net. The dual of N is another FA $TCPN$.*

Proof. Notice that in the dual, a Fork transition is translated into Attributions and vice versa. Other places and transitions are translated into single input single output transitions and places. □

Moreover, duality means the following. Every chain of places and transitions (see figure 3) is translated into another chain of places and transitions. If one transition t_i is controllable in the primal net, then its respective place p_i is measurable in the dual net. Thus the t_i downstream flow can be manipulated by controlling t_i in the primal net. In the dual net the p_i upstream marking can be inferred by measuring p_i.

3 Controllability in FA nets

In this section the loss of controllability in FA nets, due to the presence of forks and upstream transitions having equal rates, is explained. Let us firstly introduce the following controllability condition.

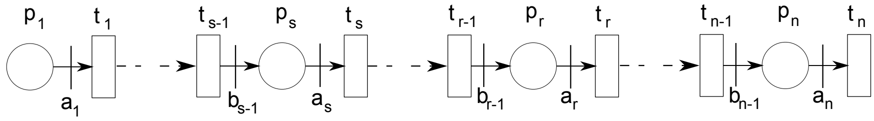


Figure 1: A place-transition chain.

Proposition 2. *Consider a FA-TCPN, if there exist a vector* $\mathbf{b} \neq \mathbf{0}$ *such that* $b_i = 0 \ \forall t_i \in T_c$ *and* $\mathbf{b}^T\mathbf{f}(\tau) = 0$ *for all* τ*, then, the rank of the controllability matrix of the TCPN is less than* $|P|$.

Proof. Using the solution of the state equation, it can be shown that (see proof of lemma 5.5 in Vázquez et. al. (2013))

$$\mathbf{m}(\tau) = Cont \cdot \begin{bmatrix} g_1(u,\tau) \\ \vdots \\ g_n(u,\tau) \end{bmatrix}$$

where the $g_i(u,\tau)$ are linearly independent functions of time and the input u. Furthermore $\mathbf{f}(\tau) = \mathbf{\Lambda\Pi m}(\tau)$, thus, $(\mathbf{b}^T\mathbf{\Lambda\Pi}) \cdot Cont = \mathbf{0}$, then $(\mathbf{b}^T\mathbf{\Lambda\Pi})$ is a left annuler of $Cont$. □

A vector $\mathbf{b}$ that fulfils the conditions of the previous proposition appears when linear dependencies between flows of uncontrollable transitions hold ($b_i = 0 \ \forall t_i \in T_c$ and $\mathbf{b}^T\mathbf{f}(\tau) = 0$). A case of such linear dependencies appears when a fork exists and the rates of some downstream transitions are equal. This result is formally stated in Proposition 5. In order to prove such proposition, a couple of preliminarily results are required (Propositions 3 and 4).

Proposition 3. *Consider a place-transition chain with* n *places and* n *transitions as in fig. 1. Suppose that the indexes of places and transitions are such that* $\{p_j\} = ^\bullet t_j$ *and* $t_j^\bullet = \{p_{j+1}\}$ *for all index* j*. Consider the Laplace transform of flows of arbitrary transitions* t_r *and* t_s*, with* t_s *being upstream of* t_r *(i.e.,* $r < s$*) and* $\lambda_r \neq \lambda_s$*, and denote them as* $F_r(s)$ *and* $F_s(s)$*, respectively. The following equality holds*

$$\frac{F_s(s)}{s+\lambda_r} = \frac{F_s(s)}{\lambda_r - \lambda_s} - \frac{\lambda_s \alpha_s^{s-1}}{\lambda_r - \lambda_s}\frac{F_{s-1}(s)}{s+\lambda_r} \tag{2}$$

where

$$\alpha_s^{s-1} = \frac{Post(t_{s-1}, p_s)}{Pre(t_s, p_s)} \tag{3}$$

Proposition 4. *Consider a place-transition chain with* n *places and* n *transitions as in fig. 1. Suppose that the indexes of places and transitions are such that* $\{p_j\} = ^\bullet t_j$ *and* $t_j^\bullet = \{p_{j+1}\}$ *for all index* j*. Consider a transition* t_{n-r} *being upstream of* t_n *(i.e.,* $n > (n-r)$*). Suppose the rates of transitions* t_j*, with* $n > j \geq n-r$*, are different to* λ_n*. The following equality, involving the flows of the transitions from* t_n *to* t_{n-r}*, holds.*

$$f_n(\tau) = \sum_{i=1}^{r-1} \beta_{n-i} f_{n-i}(\tau) + \beta_{n-r} \int_0^\tau e^{-\lambda_n \eta} f_{n-r}(\tau - \eta) d\eta \tag{4}$$

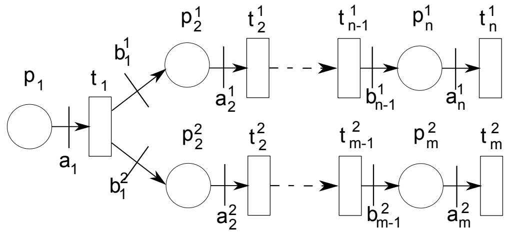


Figure 2: A fork net consisting of two place-transition chains.

where

$$\begin{aligned} \beta_{n-i} &= \prod_{j=1}^{i} \frac{\lambda_{n-j+1}\alpha_{n-j+1}^{n-j}(-1)^{i+1}}{\lambda_n - \lambda_{n-j}} \qquad \textit{for } i \leq r-1 \\ \beta_{n-r} &= \lambda_n \alpha_n^{n-1}(-1)^{r-1} \prod_{j=1}^{r-1} \frac{\lambda_{n-j}\alpha_{n-j}^{n-j-1}}{\lambda_n - \lambda_{n-j}} \end{aligned} \tag{5}$$

Proposition 5. *Consider the net of fig. 2 where transition t_1 is a fork that splits in two branches. Define the set $S = T \setminus \{t_1, t_n^1, t_m^2\}$, and suppose $\lambda_n^1 \neq \lambda(t)$, and $\lambda_m^2 \neq \lambda(t),\ \forall t \in S$.*
The flows of transitions $\{t_2^1, .., t_n^1, t_2^2, ..., t_m^2\}$ are linearly dependent iff $\lambda_n^1 = \lambda_m^2$. Furthermore such linear combination can be calculated as $\mathbf{b}^T\mathbf{f}(\tau) = \mathbf{0}$, where

$$\mathbf{b} = \left[0, -\frac{\beta_2^1}{\beta_1^1}, \ldots, -\frac{\beta_{n-1}^1}{\beta_1^1}, \frac{1}{\beta_1^1}, \frac{\beta_2^2}{\beta_1^2}, \ldots, \frac{\beta_{m-1}^2}{\beta_1^2}, -\frac{1}{\beta_1^2}\right]^T$$

$$\mathbf{f}(\tau) = \left[f_1^1(\tau), f_2^1(\tau), \ldots, f_n^1(\tau), f_2^2(\tau), \ldots, f_m^2(\tau)\right]^T$$

and constants β_1^j (resp. β_2^j) are calculated as in equation (5) with $r = n-1$ (resp. $r = m-1$).

Proof. Sufficiency. By Proposition 4, the flows for the first chain can be represented as

$$f_n^1(\tau) = \sum_{i=1}^{n-2} \beta_{n-i}^1 f_{n-i}^1(\tau) + \beta_1^1 \int_0^\tau e^{-\lambda_n^1 \eta} f_1^1(\tau - \eta) d\eta$$

where the superindex 1 in the constants β means that such values are computed as in (5) using the nodes in the chain one. Equivalently,

$$\int_0^\tau e^{-\lambda_n^1 \eta} f_1^1(\tau - \eta) d\eta = \frac{1}{\beta_1^1} f_n^1(\tau) - \sum_{i=1}^{n-2} \frac{\beta_{n-i}^1}{\beta_1^1} f_{n-i}^1(\tau)$$

Similarly, the flows for the second chain can be represented as

$$\int_0^\tau e^{-\lambda_m^2 \eta} f_1^2(\tau - \eta) d\eta = \frac{1}{\beta_1^2} f_m^2(\tau) - \sum_{i=1}^{m-2} \frac{\beta_{m-i}^2}{\beta_1^2} f_{m-i}^2(\tau)$$

Since t_1^1 and t_1^2 are actually the same transition, the left terms of previous equation are equal if $\lambda_n^1 = \lambda_m^2$. In such case, equalling the right terms of both equations it follows

$$\frac{1}{\beta_1^1} f_n^1(\tau) - \sum_{i=1}^{n-2} \frac{\beta_{n-i}^1}{\beta_1^1} f_{n-i}^1(\tau) = \frac{1}{\beta_1^2} f_m^2(\tau) - \sum_{i=1}^{m-2} \frac{\beta_{m-i}^2}{\beta_1^2} f_{m-i}^2(\tau)$$

Note from (5) that the coefficients are not null. Then, a linear combination involving the flows of transitions $\{t_2^1, .., t_n^1, t_2^2, ..., t_m^2\}$ holds. Furthermore last ecuation is equivalent to $\mathbf{b}^T\mathbf{f}(\tau) = \mathbf{0}$.

Necessity. Let us proceed by contradiction. Suppose that the rates of all the transitions are different but there is a linear combination involving the flows of the mentioned transitions, then, for some coefficients a_k^1 and a_k^2 being not all null, it holds

$$\sum_{k=2}^{n} a_k^1 f_k^1(\tau) = \sum_{k=2}^{m} a_k^2 f_k^2(\tau)$$

By using Proposition (4) in order to represent the flow of each transition $f_k^1(\tau)$ and $f_k^2(\tau)$ as a weighted sum of the flows of upstream transitions and the convolution of $f_1(\tau)$, and substituting in the previous equation, it is obtained

$$\sum_{k=2}^{n} a_k^1 \left[\sum_{j=1}^{k-2} \beta_{k,j}^1 f_{k-j}^1(\tau) + \beta_{k,1}^1 \int_0^\tau e^{-\lambda_k^1 \eta} f_1^1(\tau - \eta) d\eta\right] =$$
$$\sum_{k=2}^{m} a_k^2 \left[\sum_{j=1}^{k-2} \beta_{k,j}^1 f_{k-j}^1(\tau) + \beta_{k,1}^1 \int_0^\tau e^{-\lambda_k^2 \eta} f_1^1(\tau - \eta) d\eta\right]$$

where $\beta_{k,j}^1$ (resp. $\beta_{k,j}^2$) is computed as in (5) using k instead of n and j instead of i with the weights and rates of the first (resp. second) chain. Note that, according to (5), $\beta_{k,1}^1$ and $\beta_{k,1}^2$ are not null. Moreover, since the rates of all the transitions are different (by contradiction hypothesis) then the exponent functions in the convolution integrals are all linear independent functions of time. Thus, for the previous equality to hold it is required that all the coefficients a_k^1 and a_k^2 be null, leading thus to a contradiction. □

Remark. Following a similar procedure, previous proposition can be generalized to consider the case in which the rate λ_n^1 is equal to the rate of an upstream transition or more than two chains exist. In such case, a linear combination appears for every pair of transitions (t_n^1, t_m^2) having the same rates. Moreover, when the flows of transitions $\{t_2^1, .., t_n^1, t_2^2, ..., t_m^2\}$ are linearly dependent, then there exists a vector $\mathbf{b}$, whose entries (excepting b_1) are not null, s.t. $\mathbf{b}^T\mathbf{f}(\tau) = \mathbf{0}$, for all τ. In this case, there exists an invariant marking set such that the control actions do not act over it.

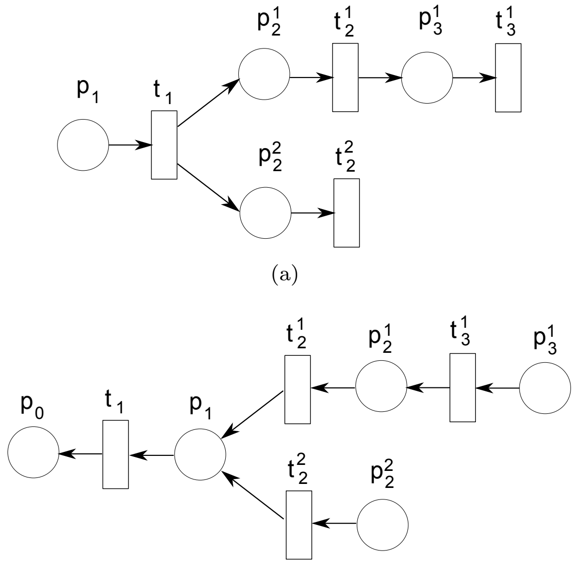
Example. Consider the net of fig. 3(a) with $\boldsymbol{\lambda} = [\lambda_1\ \lambda_2^1\ \lambda_3^1\ \lambda_2^2]$, and suppose $\lambda_3^1 = \lambda_2^2$. Then, from the proof of proposition 5,

$$\begin{bmatrix}\beta_2^1 & \beta_3^1 & \beta_2^2\end{bmatrix} = \begin{bmatrix}\frac{\lambda_2^1\lambda_3^1}{\lambda_2^1-\lambda_3^1} & \frac{-\lambda_3^1}{\lambda_2^1-\lambda_3^1} & \lambda_2^2\end{bmatrix}$$

and the components of the vector $\mathbf{b} = \begin{bmatrix}0 & \frac{\beta_3^1}{\beta_2^1} & \frac{-1}{\beta_2^1} & \frac{1}{\beta_2^2}\end{bmatrix}$ are the coefficients that form the linear combination of the flows of the transitions such that $\mathbf{b}^T\mathbf{f}(\tau) = \mathbf{0}$, for all τ. Moreover, by proposition 2

$$\mathbf{b}^T\boldsymbol{\Lambda\Pi} = \begin{bmatrix}0 & \frac{\lambda_2^1\beta_3^1}{\beta_2^1} & \frac{-\lambda_3^1}{\beta_2^1} & \frac{\lambda_2^2}{\beta_2^2}\end{bmatrix}$$

is an annuler of the controllability matrix of the $TCPN$.



(a)

(b)

Figure 3: (a) A FA net, and (b) its dual.

4 Duality

In this section, the relationship between controllability and observability matrices of primal and dual nets are analysed. Next theorem states how these matrices can be built, after that, the relationship between ranks is stated.

Theorem 6. *Let $\langle \mathcal{N}, \mathbf{\Lambda} \rangle$ be Fork-Attribution TCPN. Without loose of generality, suppose the transition t_i is being constrained by the place p_i, for each $t_i \in T$. Define the dual fluid net as $\langle \mathcal{N}^T, \mathbf{\Lambda} \rangle$. Furthermore, define a set of controllable transitions in the dual net as T_c, such that t_i is controllable in the dual net iff p_i is measured in the primal. Then, the observability matrix of the primal TCPN $\langle \mathcal{N}, \mathbf{\Lambda} \rangle$ (here denoted as Obs_{primal}) is related to the controllability matrix of the dual TCPN $\langle \mathcal{N}^T, \mathbf{\Lambda} \rangle$ (here denoted as $Cont_{dual}$) as follows*

$$\begin{bmatrix} Obs_{primal} \\ \mathbf{O}^T \cdot (\mathbf{C\Lambda\Pi})^n \end{bmatrix}^T = [\mathbf{O}, \mathbf{\Pi\Lambda} \cdot Cont_{dual}] \tag{6}$$

Proof. By definition of the observability matrix for the primal net, the left term of (6) is

$$\begin{bmatrix} Obs_{primal} \\ \mathbf{O}^T \cdot (\mathbf{C\Lambda\Pi})^n \end{bmatrix} = \begin{bmatrix} \mathbf{O}^T \\ \mathbf{O}^T \cdot \mathbf{C\Lambda\Pi} \\ \vdots \\ \mathbf{O}^T \cdot (\mathbf{C\Lambda\Pi})^{n-1} \\ \mathbf{O}^T \cdot (\mathbf{C\Lambda\Pi})^n \end{bmatrix}$$

Next, supposing transition t_i is being constrained by the place p_i, for each $t_i \in T$, it holds that $\mathbf{\Pi}$ is a diagonal matrix. Applying this to the previous equation and transposing it, it follows

$$\begin{array}{c}\left[\begin{array}{c} Obs_{primal} \\ \mathbf{O}^T \cdot (\mathbf{C\Lambda\Pi})^n \end{array}\right]^T = \left[\mathbf{O}, \mathbf{\Pi\Lambda C}^T \cdot \mathbf{O}..., (\mathbf{\Pi\Lambda C}^T)^n \cdot \mathbf{O}\right] \\ = \left[\mathbf{O}, \mathbf{\Pi\Lambda} \cdot \left(\mathbf{C}^T\mathbf{O}\right), ..., \mathbf{\Pi\Lambda} \cdot (\mathbf{C}^T\mathbf{\Pi\Lambda})^{n-1} \cdot \left(\mathbf{C}^T\mathbf{O}\right)\right]\end{array} \tag{7}$$

On the other hand, considering $||\mathbf{O}|| = T_c$ it follows that $\left(\mathbf{C}^T\mathbf{O}\right) = \mathbf{C}^T[T_c]$. Furthermore, for the same assumption that p_i constrains t_i in the primal net, p_i constrains t_i in the dual net, thus $\mathbf{\Pi}_{dual} = \mathbf{\Pi}$. Thus, the controllability matrix for the dual net can be represented as

$$\begin{array}{l} Cont_{dual} = \\ \left[\left(\mathbf{C}^T\mathbf{O}\right), (\mathbf{C}^T\mathbf{\Lambda\Pi}) \cdot \left(\mathbf{C}^T\mathbf{O}\right), ..., (\mathbf{C}^T\mathbf{\Lambda\Pi})^{n-1} \cdot \left(\mathbf{C}^T\mathbf{O}\right)\right]\end{array} \tag{8}$$

Comparing this equation with (7), and being $\mathbf{\Lambda\Pi} = \mathbf{\Pi\Lambda}$ since both $\mathbf{\Pi}$, $\mathbf{\Lambda}$ are diagonal matrix, it is easy to see that (6) holds.

□

Corollary 7. *Consider a primal ordinary Fork-Attribution net and its dual as defined in Theorem 6. Then,*

$$rank(Obs_{primal}) = rank([\mathbf{O}, Cont_{dual}]) \tag{9}$$

Proof. Premultiplying (6) by $(\mathbf{\Pi\Lambda})^{-1}$. Since $(\mathbf{\Pi\Lambda})^{-1}$ is a full rank matrix, it follows

$$rank\left(\left[\begin{array}{c} Obs_{primal} \\ \mathbf{O}^T(\mathbf{C\Lambda\Pi})^n \end{array}\right]\right) = rank\left(\left[(\mathbf{\Pi\Lambda})^{-1}\mathbf{O}, Cont_{dual}\right]\right)$$

By the Cayley-Hamilton theorem, the left term is equal to $rank(Obs_{primal})$. Finally, since $\mathbf{O}$ is a matrix whose columns are elementary vectors, and $\mathbf{\Pi\Lambda}$ is a full rank diagonal matrix, the right term of the previous equation is equal to $rank([\mathbf{O}, Cont_{dual}])$ thus (9) follows. □

Remark. Corollary 7 shows that duality between controllability and observability does not hold as in LTI systems, where the primal is observable (resp. controllable) iff the dual is controllable (resp. observable). In $TCPN$s, even if the controllability matrix of the dual does not have full rank, the observability matrix may still have full rank, because the term $\mathbf{O}$ in (9) may increase the observability rank.

Even if controllability and observability are not exactly dual properties, the relation provided by Theorem 6 can be used to extend results from controllability to observability and viceversa. Next section presents how proposition 5 can be translated to observability in FA nets.

5 Observability in FA nets

Proposition 5 regarding figure 2 shows that when there exist a fork and two downstream transitions with equal flow rates $\lambda_n^1 = \lambda_m^2$, then the flows of transitions $\{t_2^1, .., t_n^1, t_2^2, ..., t_m^2\}$ are linearly dependent. In this case there exists a vector $\mathbf{b}$, whose entries (excepting b_1) are not null, s.t. $\mathbf{b}^T\mathbf{f}(\tau) = \mathbf{0}$, for all τ. It means that there exists an invariant marking set such that the control

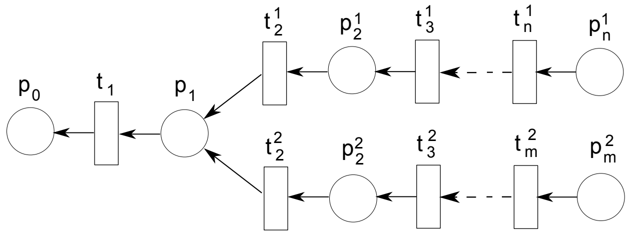


Figure 4: An attribution net consisting of two place-transition chains.

actions cannot be modified. Now, using the results presented in section 4 the dual result will be proved, meaning that an invariant marking set exists in the dual net (induced by vector $\mathbf{b}$) such that the sensors (located downstream from the attribution place) cannot infer the marking in this invariant.

Proposition 8. *Consider the net of fig. 4 where place p_1 is an attribution that combines two branches. Define the set $S = T \setminus \{t_1, t_n^1, t_m^2\}$, and suppose $\lambda_n^1 \neq \lambda(t)$, and $\lambda_m^2 \neq \lambda(t)$, $\forall t \in S$. Let p_0 be the only measured place.*
There exists a vector $\mathbf{b} \neq \mathbf{0}$ in the right kernel of the observability matrix (consequently, the net is not observable) iff $\lambda_n^1 = \lambda_m^2$.

Proof. The dual net of fig. 4, without the measured place p_0, is the net of fig. 2. Let us focus on the dual net without p_0. According to Proposition 5, there is a linear dependency involving the flows of transitions $\{t_2^1, .., t_n^1, t_2^2, ..., t_m^2\}$ in fig. 2 iff $\lambda_n^1 = \lambda_m^2$. In such case, there would exist a vector $\mathbf{b}$, whose entries (excepting b_1) are not null, s.t. $\mathbf{b}^T \mathbf{f}(\tau) = \mathbf{0}$, for all τ. Moreover by proposition 2 $\mathbf{b}^T \mathbf{\Pi\Lambda} Cont_{dual} = \mathbf{b}^T \mathbf{\Lambda\Pi} Cont_{dual} = \mathbf{0}$. Then, by defining $\mathbf{b}' = [0, \mathbf{b}^T]^T$, it follows that $\mathbf{b}'^T \mathbf{\Pi}' \mathbf{\Lambda}' Cont'_{dual} = \mathbf{0}$, where $Cont'_{dual}$ is the controllability matrix of the dual net with the place p_0. Furthermore, since p_0 is the only measured place (in the primal), then $\mathbf{b}'^T \mathbf{O} = \mathbf{0}$. Therefore, according to Theorem 6, $Obs_{primal} \cdot \mathbf{b}' = \mathbf{0}$, consequently the net becomes unobservable. □

Remark. Previous proposition determines that the dimension of the observable space can be reduced by attributions when the rates of upstream transitions from different branches are equal (already shown by Silva et al. (2011); Aguayo et al. (2011)). It also shows that the marking in places belonging to the support of vector $\mathbf{b}$ belong to an invariant subspace of the observability matrix. This invariant belongs to the kernel of the output matrix. Notice that this observability property was derived as the dual of controllability provided by Proposition 5, where the dimension of the controllable space can be reduced by forks (the dual element of attribution) when the rates of downstream transitions from different branches are equal.

Example. Consider the net of fig. 3(b) with $\boldsymbol{\lambda} = [\lambda_1\ \lambda_2^1\ \lambda_3^1\ \lambda_2^2]$; the dual of this net without place p_0 is the net of fig. 3(a) with the same $\boldsymbol{\lambda}$. Then, if $\lambda_3^1 = \lambda_2^2$, there exists a vector $\mathbf{b} \neq \mathbf{0}$ in the right kernel of the observability matrix, calculated from example of section 3 as

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & \frac{\beta_3^1}{\beta_2^1} & \frac{-1}{\beta_2^1} & \frac{1}{\beta_2^2} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & \frac{-1}{\lambda_2^1} & \frac{\lambda_3^1 - \lambda_2^1}{\lambda_2^1 \lambda_3^1} & \frac{1}{\lambda_2^2} \end{bmatrix}^T$$

and the net is not observable for that $\boldsymbol{\lambda}$.

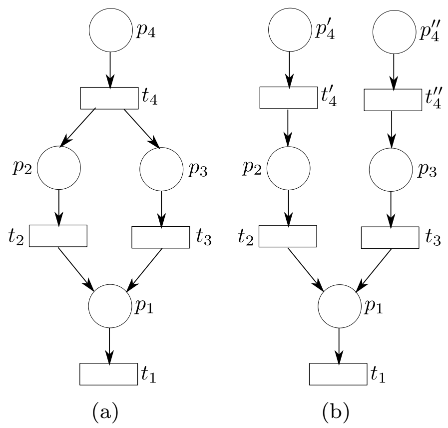


Figure 5: (a) A FA net, and (b) its equivalent attribution net.

Previous proposition does not explain the loss of observability in other cases.

Example. Consider the net of fig. 5(a) with $\boldsymbol{\lambda} = [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4]$; the dual of this net has the same graph with different labelling and rates; this fork-attribution structure is known that it introduces unobservable subspaces (Mahulea et al. (2005)). For instance, measuring p_1 the net is not observable for a specific $\boldsymbol{\lambda}$ even with $\lambda_2 \neq \lambda_3$. This can be explained using the net in fig. 5(b). If the rates of transitions t'_4 and t''_4 are defined equal to λ_4 of the net in fig. 5(a), then both nets describe the same evolution when $\mathbf{m}[p'_4] = \mathbf{m}[p''_4] = \mathbf{m}[p_4]$. Since rates of two transition are equal, following proposition 8 a vector in the kernel of the observability matrix is obtained as

$$\mathbf{b} = \begin{bmatrix} 0 & \frac{-1}{\lambda_2} & \frac{1}{\lambda_3} & -\frac{\lambda_2-\lambda_4}{\lambda_4\lambda_2} & \frac{\lambda_3-\lambda_4}{\lambda_4\lambda_3} \end{bmatrix}^T$$

then the net in fig. 5(b) is not observable in this invariant, then the net in fig. 5(a) is unobservable when the subspace $\mathbf{m}[p'_4] = \mathbf{m}[p''_4]$ is inside this invariant, that is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 \end{bmatrix} \cdot \mathbf{b} = 0$$

This holds for $\lambda_4 = \frac{2\lambda_2\lambda_3}{\lambda_2+\lambda_3}$, which corresponds with the rate at which the net in fig. 5(a) is not observable (in accordance to Mahulea et al. (2005)).

6 Conclusions

This work uses duality property to relate controllability and observability in FA nets. This class of nets is closed under duality transformations leading to an explanation of the main relationship between controllability and observability in dual nets. In order to show the application of duality, we extended the results of controllability in FA nets in section 4 and translated, by duality, into the observability framework (5). In the future this work will be extended to other classes of PN, for example, it can be extended to non conflict-free PN using a transformation presented in other works.

References

Aguayo-Lara, E., Ramírez-Treviño, A. and Ruiz-León, J. (2011), "Invariant subspaces and sensor placement for observability in continuous timed Petri nets ", In *IEEE Conference on Automation Science and Engineering*, Trieste, Italy, pp. 607-612.

Chi-Tsong Chen (1984),"Linear Systems Theory and Design", *Saubders College Publishing, Philadelphia, PA, USA*

David, R., and Alla, H. (2010), *Discrete, continuous and hybrid Petri nets*, Springer, 2nd ed.

Hack, M. (1972),"Analysis of production schemata by Petri nets", *TR 94, Project MAC, Massachusetts Inst. of Technology.*

Júlvez, J. and Boel, R.K. (2010), "A continuous Petri net approach for Model Predictive Control of traffic systems", *IEEE Transactions on Systems, Man and Cybernetics Part A*, vol. 40, no. 4, pp. 686-697.

Kara, R., Ahmane, M., Loiseau, J.J., and Djennoune, S. (2009), "Constrained regulation of continuous Petri nets ", *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 738–748.

Lefebvre, D. (1999), "Feedback control designs for manufacturing systems modelled by continuous Petri nets, ", *International Journal of Systems Science*, vol. 30, no. 6, pp. 591-600.

Mahulea, C., Recalde, L., and Silva, M. (2005) , "Optimal Observability For Continuous Petri Nets ", *In 16 IFAC World Congress*, pp. 65.

Mahulea, C., Giua, A., Recalde, L., Seatzu, C., and Silva, M. (2008), "Optimal model predictive control of Timed Continuous Petri nets ", *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1731–1735.

Mahulea, C., Ramírez-Treviño, A., Recalde, L. and Silva, M. (2008), "Steady state control reference and token conservation laws in continuous Petri net system ", *IEEE Trans. on Automation Science and Engineering*, vol. 2, no. 5, pp. 726-741.

Ross-León, R., Ramirez-Treviño, A., Morales, J. A., and Ruiz-León, J. (2010), "Control of Metabolic Systems Modeled with Timed Continuous Petri Nets ", *International Workshop on Biological Processes & Petri Nets*, pp. 85-100.

Silva, M. (1993), *Introducing Petri nets, In Practice of Petri Nets in Manufacturing*, Chapman & Hall.

Silva, M., Júlvez, J., Mahulea, C. and Vázquez, C.R. (2011), "On fluidization of discrete event models: observation and control of continuous Petri nets ", *Discrete Event Dynamic Systems*, vol. 21, no. 4, pp. 427-497.

Vázquez, C.R., and Ramírez-Treviño, A. (2012) , "Structural and generic conditions for controllability of TCPNs ", *In Proc. of the 11th International Workshop on Discrete Event Systems*, Guadalajara, Mexico.

Vázquez, C.R., Ramírez, A., and Silva, M. (2013), "Controllability of timed continuous Petri nets with uncontrollable transitions ", *International Journal of Control*, DOI:10.1080/00207179.2013.846480

A Flow linear dependencies

Proposition 9. *Consider a place-transition chain with n places and n transitions as in fig. 1. Suppose that the indexes of places and transitions are such that $\{p_j\} = ^\bullet t_j$ and $t_j^\bullet = \{p_{j+1}\}$ for all index j. Consider the Laplace transform of flows of arbitrary transitions t_r and t_s, with t_s being upstream of t_r (i.e., $r < s$) and $\lambda_r \neq \lambda_s$, and denote them as $F_r(s)$ and $F_s(s)$, respectively. The following equality holds*

$$\frac{F_s(s)}{s+\lambda_r} = \frac{F_s(s)}{\lambda_r - \lambda_s} - \frac{\lambda_s \alpha_s^{s-1}}{\lambda_r - \lambda_s} \frac{F_{s-1}(s)}{s+\lambda_r} \tag{10}$$

where

$$\alpha_s^{s-1} = \frac{Post(t_{s-1}, p_s)}{Pre(t_s, p_s)} \tag{11}$$

Proof. Let us firstly derive a transfer function for consecutive transitions' flows. Given an arbitrary transition t_n, its flow is given by $f_n(\tau) = \lambda_n enab(t_n) = \lambda_n \frac{m_n(\tau)}{Pre(t_n,p_n)}$. Now, $\dot{m}_n(\tau) = Pos(t_{n-1}, p_n) f_{n-1}(\tau) - Pre(t_n, p_n) f_n(\tau)$. Combining these equations it follows

$$\dot{f}_n(\tau) = \lambda_n \left[\frac{Pos(t_{n-1}, p_n)}{Pre(t_n, p_n)} f_{n-1}(\tau) - f_n(\tau) \right]$$

Taking the Laplace transform and using the notation (3), previous equation leads to $sF_n(s) = \lambda_n \alpha_n^{n-1} F_{n-1}(s) - \lambda_n F_n(s)$. Reordering the terms, we obtain the transfer function for the flow at t_n taking as input the flow of the upstream transition t_{n-1} as

$$\frac{F_n(s)}{F_{n-1}(s)} = \frac{\lambda_n \alpha_n^{n-1}}{s+\lambda_n} \tag{12}$$

Now, let us proceed with the proof of (2). By using the previous equation with s instead of n, it can be stated

$$\frac{F_r(s)}{F_{r-1}(s)} F_s(s) = \frac{F_r(s)}{F_{r-1}(s)} \left(\frac{\lambda_s \alpha_s^{s-1}}{s+\lambda_s} F_{s-1}(s) \right)$$

Adding and resting an auxiliary term $\frac{\lambda_s \alpha_s^{s-1}}{\lambda_r - \lambda_s} \frac{F_r(s)}{F_{r-1}(s)} F_{s-1}(s)$, reordering terms and using (12) with r instead of n, it follows

$$\begin{aligned} \tfrac{F_r(s)}{F_{r-1}(s)} F_s(s) = &-\tfrac{\lambda_s \alpha_s^{s-1}}{\lambda_r - \lambda_s} \tfrac{F_r(s)}{F_{r-1}(s)} F_{s-1}(s) \\ &+ \left(\tfrac{\lambda_r \alpha_r^{r-1} \lambda_s \alpha_s^{s-1}}{(s+\lambda_r)(s+\lambda_s)} + \tfrac{\lambda_r \alpha_r^{r-1} \lambda_s \alpha_s^{s-1}}{(s+\lambda_r)(\lambda_r - \lambda_s)} \right) F_{s-1}(s) \end{aligned}$$

It is easy to shown that the term in parenthesis is equal to $\frac{\lambda_r \alpha_r^{r-1} \lambda_s \alpha_s^{s-1}}{(s+\lambda_s)(\lambda_r - \lambda_s)}$. Substituting this term, it holds

$$\begin{aligned} \tfrac{F_r(s)}{F_{r-1}(s)} F_s(s) = &-\tfrac{\lambda_s \alpha_s^{s-1}}{\lambda_r - \lambda_s} \tfrac{F_r(s)}{F_{r-1}(s)} F_{s-1}(s) \\ &+ \tfrac{\lambda_r \alpha_r^{r-1} \lambda_s \alpha_s^{s-1}}{(s+\lambda_s)(\lambda_r - \lambda_s)} F_{s-1}(s) \end{aligned}$$

The last term of previous equation can be simplified by using (12) with s instead of n. Thus,

$$\frac{F_r(s)}{F_{r-1}(s)}F_s(s) = -\frac{\lambda_s\alpha_s^{s-1}}{\lambda_r-\lambda_s}\frac{F_r(s)}{F_{r-1}(s)}F_{s-1}(s) + \frac{\lambda_r\alpha_r^{r-1}}{\lambda_r-\lambda_s}F_s(s)$$

Finally, (2) is obtained by substituting the term $\frac{F_r(s)}{F_{r-1}(s)}$ by (12) with r instead of n and dividing the whole equation by $\lambda_r\alpha_r^{r-1}$. □

Proposition 10. *Consider a place-transition chain with n places and n transitions as in fig. 1. Suppose that the indexes of places and transitions are such that $\{p_j\} = {}^\bullet t_j$ and $t_j^\bullet = \{p_{j+1}\}$ for all index j. Consider a transitions t_{n-r} being upstream of t_n (i.e., $n > (n-r)$). Suppose the rates of transitions t_j, with $n < j \geq n-r$, are different to λ_n. The following equality, involving the flows of the transitions from t_n to t_{n-r}, holds.*

$$f_n(\tau) = \sum_{i=1}^{r-1}\beta_{n-i}f_{n-i}(\tau) + \beta_{n-r}\int_0^\tau e^{-\lambda_n\eta}f_{n-r}(\tau-\eta)d\eta \tag{13}$$

where

$$\begin{aligned} \beta_{n-i} &= \prod_{j=1}^{i}\frac{\lambda_{n-j+1}\alpha_{n-j+1}^{n-j}(-1)^{i+1}}{\lambda_n-\lambda_{n-j}} \qquad \textit{for } i \leq r-1 \\ \beta_{n-r} &= \lambda_n\alpha_n^{n-1}(-1)^{r-1}\prod_{j=1}^{r-1}\frac{\lambda_{n-j}\alpha_{n-j}^{n-j-1}}{\lambda_n-\lambda_{n-j}} \end{aligned} \tag{14}$$

Proof. From (12) it is known that $F_n(s) = \frac{\lambda_n\alpha_n^{n-1}}{s+\lambda_n}F_{n-1}(s)$. Now, an expression for $\frac{F_{n-1}(s)}{s+\lambda_n}$ can be obtained by using (2) with n instead of r and $n-1$ instead of s. Substituting such term into previous equation, it follows

$$F_n(s) = \frac{\lambda_n\alpha_n^{n-1}}{\lambda_n-\lambda_{n-1}}\left[F_{n-1}(s) - \frac{\lambda_{n-1}\alpha_{n-1}^{n-2}}{s+\lambda_n}F_{n-2}(s)\right]$$

Next, an expression for $\frac{F_{n-2}(s)}{s+\lambda_n}$ can be obtained by using (2) with n instead of r and $n-2$ instead of s. Substituting such term into previous equation, it follows

$$F_n(s) = \frac{\lambda_n\alpha_n^{n-1}}{\lambda_n-\lambda_{n-1}}\left[F_{n-1}(s) - \frac{\lambda_{n-1}\alpha_{n-1}^{n-2}}{\lambda_n-\lambda_{n-2}}\cdot \cdot\left(F_{n-2}(s) - \frac{\lambda_{n-2}\alpha_{n-2}^{n-3}}{s+\lambda_n}F_{n-3}(s)\right)\right]$$

By iterating this substitution procedure, using (2) with n instead of r and $n-j$ instead of s for j from 3 until $r+1$, the following expression is obtained

$$F_n(\tau) = \sum_{i=1}^{r-1}\beta_{n-i}F_{n-i}(\tau) + \beta_{n-r}\frac{F_{n-r}(s)}{s+\lambda_n}$$

where coefficients β are given by (5). Finally, (4) is obtained by applying the inverse Laplace transform to previous equation. □